ANALYSIS OF FRACTIONAL ORDER LOWPASS AND HIGHPASS FILTERS

B.T.KRISHNA[1] K.V.V.S.REDDY[2]
1. GITAM institute of technology, GITAM University, Visakhapatnam, India.
2. College of Engineering, Andhra University, Visakhapatnam, India.
Email: - tkbattula@gmail.com

Abstract: *In this paper, analysis of fractional order passive RC low-pass and high pass filter circuits is presented. The time-domain expressions for different values of fractional order,* α *were calculated using laplace transform approach. The effect of fractional order on frequency response is studied. It has been observed that the second order characteristics can be obtained from a fractional circuit of order between 1 and 2.*

Keywords**:** Fractional calculus, mittag-leffler function, fractional order circuit, magnitude response and phase response.

1. Introduction**:** Fractional calculus is a very old topic [1]. It deals with the generalization of differentiation and integration to an arbitrary order. Now a day's attention is drawn towards the use of fractional calculus in the fields of control systems, signal processing, electric circuits, Electromagnetics etc. The fractional integration of α^{th} order of the function $f(t)$ is defined as [2],

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \qquad (1)$$

The Laplace Transform of the fractional Integral is[2],

$$\text{L}\{J^{\alpha} f(t)\} = S^{-\alpha} F(S) \qquad (2)$$

where $F(S)$ is the Laplace transform of $f(t)$.The Riemann-Liouville Fractional differentiation of the function $f(t)$ of order $\alpha(\alpha > 0)$ is defined as[2],

$$aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n}\{\int_a^t (t-\tau)^{n-\alpha+1} f(\tau) d\tau\} \qquad (3)$$

$$n-1 < \alpha < n$$

where $\Gamma(x)$ represents the gamma function of x.The Laplace transform of the fractional derivative is,

$$\text{L}\{D^{\alpha} f(t)\} = S^{\alpha} F(S) \qquad (4)$$

Mittag-Leffler function: The Mittag-Leffler function plays an important role in the solution of fractional order differential equations. The Mittag-Leffler function (1903) is defined as [2],

$$E_{\alpha}(x) = \sum_{K=0}^{\infty} \frac{x^K}{\Gamma(\alpha K + 1)} \quad \alpha \geq 0 \qquad (5)$$

if $\alpha = 0, E_0(x) = \frac{1}{1-x}$ and it will be an exponential function, when the value of $\alpha = 1$.So $E_1(x) = e^x$ [7].The above function defined in eqn. (5) is generalized by wiman (1905) as [5-7],

$$E_{\alpha,\beta}(x) = \sum_{K=0}^{\infty} \frac{x^K}{\Gamma(\alpha K + \beta)} \quad \alpha \geq 0, \beta \geq 0 \qquad (6)$$

The corresponding Laplace transform pair is [12],

$$\frac{S^{\alpha-\beta}}{S^{\alpha}+\eta} \longleftrightarrow t^{\beta-1} E_{\alpha,\beta}(-\eta t^{\alpha}) \qquad (7)$$

$$If\ \beta = 1, E_{\alpha,1}(x) = E_{\alpha}(x) \qquad (8)$$

If $\alpha = \beta = 1$, then the Mittag-Leffler function defined in eqn. (6) will become an exponential function, $E_{1,1}(x) = e^x$ (9)

$$When\ \alpha = 1, \beta = 2, E_{1,2}(x) = \frac{e^x - 1}{x} \qquad (10)$$

$$For\ \alpha = 2, \beta = 1, E_{2,1}(x) = e^{\sqrt{x}} \qquad (11)$$

$$For\ \alpha = 2, \beta = 2, E_{2,2}(x) = \frac{\sinh(\sqrt{x})}{\sqrt{x}} \qquad (12)$$

$$For\alpha = 1, \beta = 3, E_{1,3}(x) = \frac{e^x - x - 1}{x^2} \qquad (13)$$

A *fractional order capacitor* is one which is defined by the following equation,[4]

$$i(t) = C^{\alpha} \frac{d^{\alpha} v(t)}{dt^{\alpha}} \qquad (14)$$

where α is the fractional order. A fractional order circuit is one which contains at least one fractional order capacitor and is defined by a fractional order differential equation. The significant advantage of fractional order circuits compared to integer order circuits is that they are characterized by memory. Fractional order systems are characterized by infinite memory, whereas it is finite for an integer order system.

In this paper, the transfer functions of fractional order RC circuits are derived, then time and frequency domain analysis is carried out in section2.Numerical simulations and conclusions are presented in section3.

2. Fractional order Circuits**:** The following are the fractional order low-pass and high-pass filter circuits.

2.1 Fractional order low pass filter

The circuit diagram for the single stage fractional order low pass filter is shown in Fig.1.

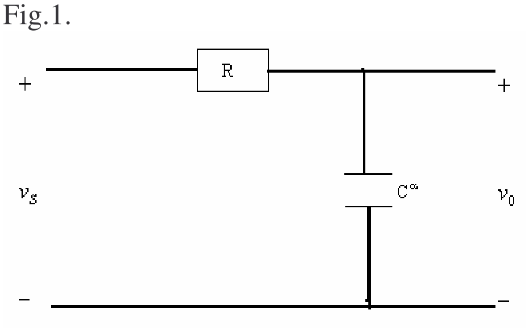

Fig.1. Fractional order low pass filter

The characteristic equation for the circuit is,

$$\frac{v_0}{R} + C^{\alpha} \frac{d^{\alpha} v_0(t)}{dt^{\alpha}} = \frac{v_S}{R} \qquad (15)$$

Taking Laplace transform, the transfer function of the circuit is

$$H(S) = \frac{V_0(S)}{V_S(S)} = \frac{\tau}{S^{\alpha} + \tau} \qquad (16)$$

Where $\tau = \frac{1}{RC^{\alpha}}$ is a constant.

The impulse response is found to be,

$$h(t) = \tau t^{\alpha - 1} E_{\alpha,\alpha}(-\tau t^{\alpha}) \qquad (17)$$

The step response is calculated as,

$$v_0(t) = 1 - E_{\alpha,1}(-\tau t^{\alpha}) \qquad (18)$$

The magnitude and phase of the transfer function are,

$$|H(\omega)| = \frac{\tau}{\sqrt{\omega^{2\alpha} + \tau^2 + 2\omega^{\alpha} \cos(\alpha\pi/2)}} \qquad (19)$$

$$\phi(\omega) = -\tan^{-1}\left[\frac{\omega^{\alpha} \sin(\alpha\pi/2)}{\tau + \omega^{\alpha} \cos(\alpha\pi/2)}\right] \qquad (20)$$

2.2 Fractional order High pass filter

The circuit diagram for the single stage fractional order High pass filter is shown in Fig.2.

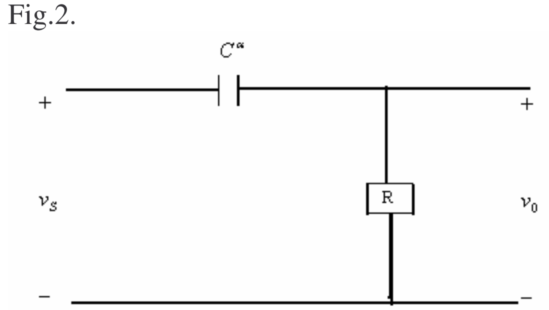

Fig.2. Fractional order High pass filter

The characteristic equation for the circuit is,

$$\frac{v_0}{R} + C^{\alpha} \frac{d^{\alpha} v_0(t)}{dt^{\alpha}} = C^{\alpha} \frac{d^{\alpha} v_S}{dt^{\alpha}} \qquad (21)$$

Taking Laplace transform, the transfer function of the circuit is

$$H(S) = \frac{V_0(S)}{V_S(S)} = \frac{S^{\alpha}}{S^{\alpha} + \tau} \qquad (22)$$

Where $\tau = \frac{1}{RC^{\alpha}}$ is a constant.

The impulse response is found to be,

$$h(t) = 1 - \tau t^{\alpha - 1} E_{\alpha,\alpha}(-\tau t^{\alpha}) \qquad (23)$$

The step response is found to be,

$$v_0(t) = E_{\alpha,1}(-\tau t^{\alpha}) \qquad (24)$$

The magnitude and phase of the transfer function are given by,

$$|H(\omega)| = \frac{\tau\omega^{\alpha}}{\sqrt{\tau^2 \omega^{2\alpha} + 1 + 2\tau\omega^{\alpha} \cos(\alpha\pi/2)}} \qquad (25)$$

$$\phi(\omega) = \tan^{-1}\left[\frac{\tau\omega^{\alpha} \sin(\alpha\pi/2)}{\tau^2 \omega^{2\alpha} + \tau\omega^{\alpha} \cos(\alpha\pi/2)}\right] \qquad (26)$$

3. Results and conclusions

3.1. Time-domain response of fractional order low-pass filter

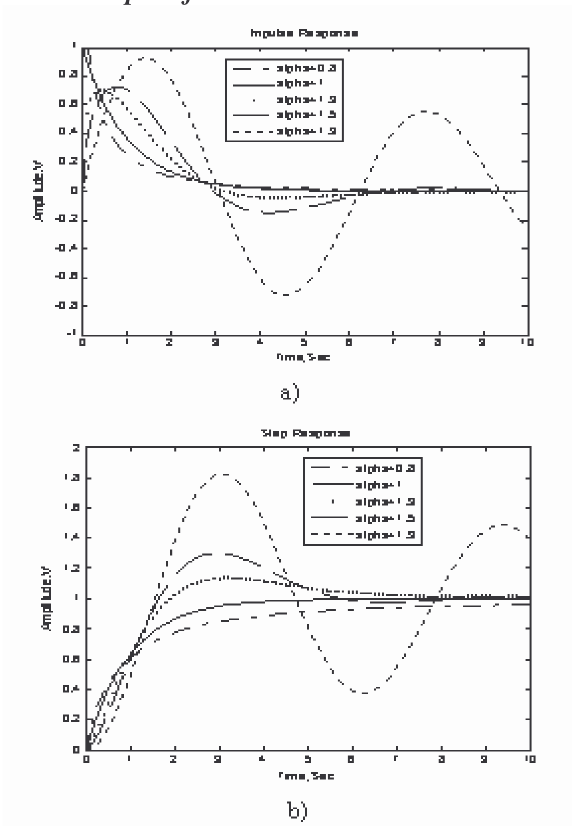

Fig.3.Time-domain response of fractional order low pass filter a) Impulse response b) Step response

3.2 frequency response of fractional order low-pass filter

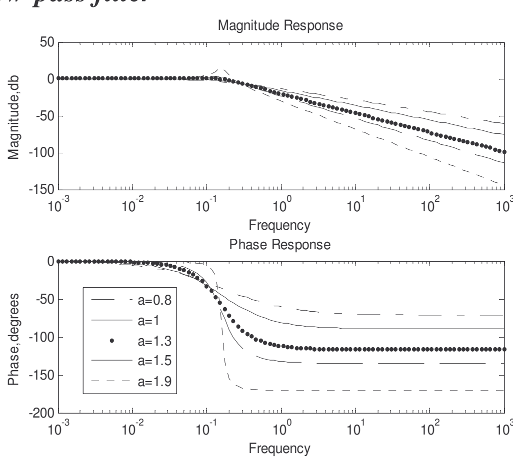

Fig.4. Magnitude and phase responses of Fractional order low-pass filter

3.3. Time-domain response of fractional order high-pass filter

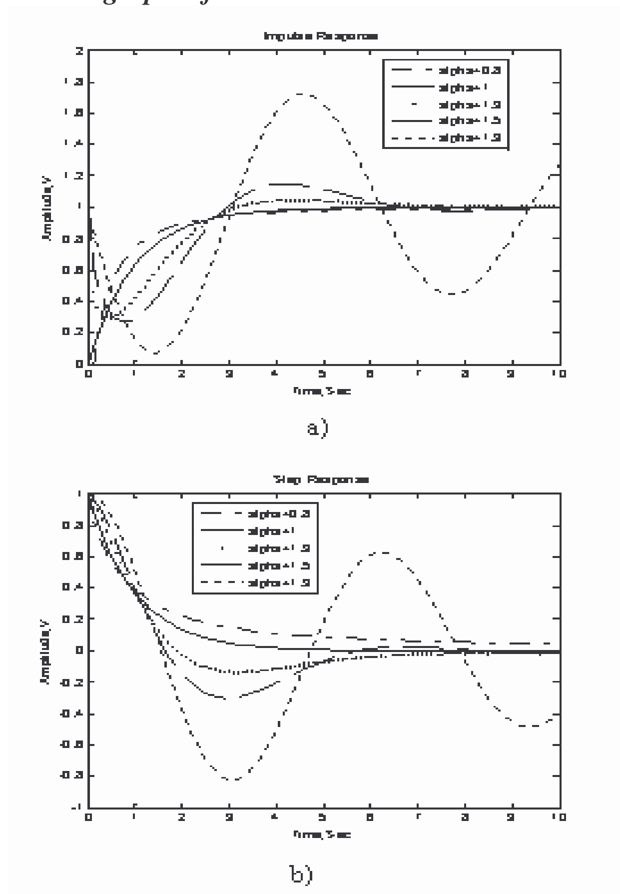

Fig.5.Time-domain response of fractional order high pass filter a) Impulse response b) Step response

3.4 Frequency response of fractional order High-pass filter

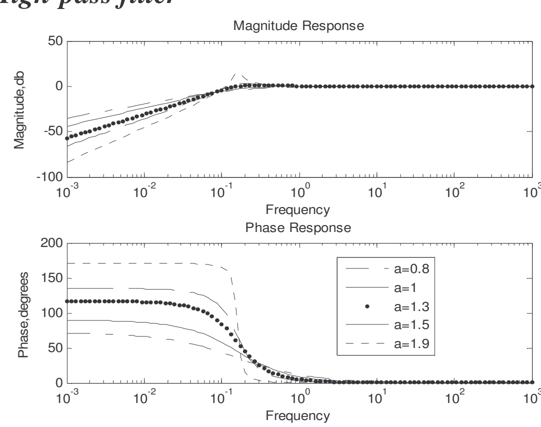

Fig.6. Magnitude and phase responses of Fractional order High-pass filter

From Figs.3, 4, 5&6, it can be observed that for lower values of α both time and frequency domain responses were inaccurate. For α >1, the magnitude plot shows a resonant peak, which increases as the order approaches 2. Similarly for α>1 the time-domain response exhibits oscillations which increase as the value of order approaches 2.The phase variations are linear over a narrow range of frequency for small values ofα.As the value of α increases the phase plot exhibits saturation behavior towards the asymptotic values.

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